International Journal of Theoretical Physics, Vol. 44, No. 12, December 2005 (© 2005)
DOI: 10.1007/s10773-005-8021-3

Residuated Semigroups and the Algebraic Foundations of Quantum Mechanics[1]

Leopoldo Román[2]

Let Q be an idempotent and right-sided quantale. There is a one to one correspondence between quantifiers and non-commutative binary operations making Q an idempotent and right-sided quantale. If Q is an atomic and irreducible orthomodular lattice there are only two such operations. Namely, the discrete quantifier and the indiscrete quantifer.

KEY WORDS: quantum logic; Orthomodular lattice; quantale; residuated semigroup.

PACS: 0210.Ab; 0210.De

1. INTRODUCTION

By the algebraic foundations of quantum mechanics, we understand the lattice theoretical approach given by Birkhkoff and von Neumann (1936). Also, we believe in the "logic" of (non-relativistic) quantum mechanics as being the lattice of closed subspaces of a separable infinite dimensional Hilbert space in the sense of Mackey (1957).

Mathematical logic and algebra have now a very close relation. Indeed, an example of the last claim is algebraic logic, a concept introduced by Halmos (1956), and in particular, the notion of a quantifer introduced in Halmos (1962) for boolean algebras. The main example in this direction is an existential quantifier.

On the other hand, Janowitz (1963) generalized the notion of a quantifier for orthomodular lattices. Orthomodular lattices have a very close relation with boolean algebras; Halmos gave a characterization of quantifiers for boolean algebras, the reader can consult Theorem 10 in Halmos (1962), for details. As far as we know, a general characterization of a quantifier for orthomodular lattices is not known. Nevertheless, if L is an atomic and

[1] This work is dedicated to Alberto Román.

[2] Institute de Matemáticas, UNAM, Area de la Investigación Científica, Ciudad Universitaria, 04510, México, D.F., Mexico; e-mail: leopoldo@matem.unam.mx.

irreducible orthomodular lattice, L admits only two quantifiers: the discrete and the indiscrete quantifiers (see Janowitz, 1963, p. 1245). The best result in finding a characterization of quantifiers is Theorem 7 in Janowitz (1963).

Quantales were introduced for a possible algebraic foundation of quantum mechanics. We shall see that idempotent, right-sided quantales are in one-to-one correspondence with closure operators which are in fact a quantifiers. Whenever mentioned, the notion of quantifiers is from a general point of view; it is not hard to define a quantifier for a bounded lattice. The concept of a quantale was introduced *only* for bounded complete lattices.

However, with the help of residuation theory we can consider a more general situation in which no completeness assumption is needed. See the book by Blyth and Janowitz (1972). We think that residuation theory is very important for the Algebraic Foundations of Quantum Mechanics. In principle, there could be a non-obvious connection between quantales and residuation theory. We shall see that this is not the case. Residuation theory is a more general framework. A quantale is just a residuated semigroup which is also a complete lattice.

The article is organized as follows. In the first section we introduce the concepts we need for our purposes. In the second section we prove that the only two non-commutative binary operations making the lattice of closed subspaces of a separable infinite dimensional Hilbert space an idempotent, right-sided quantale, are induced by the discrete and indiscrete quantifiers.

2. SECTION I

Definition 1. A quantale Q is a complete lattice together with an associative product & such that, for all $a, b \in Q$, $\{a_i\}_{i\in I}, \{b_i\}_{i\in I} \subseteq Q$:

(1) $a\&(\vee_{i\in I} b_i) = \vee_{i\in I}(a\&b_i)$;
(2) $(\vee_{i\in I} a_i)\&b = \vee_{i\in I}(a\&b)$

A quantale Q is right-sided and idempotent, if it satisfies the following two conditions:

(1) $a\&1 = a$.
(2) $a\&a = a$ for all $a \in Q$.

Remark 1.

Borceux and Bossche (1986) proved that given any complete lattice $(Q, \leq, 0, 1)$, there is a one-to-one correspondence between binary operations,

&: $Q \times Q \to Q$, making Q into an idempotent, right-sided quantale and closure operations $j : Q \to Q$ satisfying the following conditions:

(1) $a \leq j(a)$.
(2) $j(a \wedge j(b)) = j(a) \wedge j(b)$.
(3) $a \wedge j(\vee_{i \in I} b_i) = \vee_{i \in I}(a \wedge j(b_i))$.
(4) $(\vee_{i \in I} a_i) \wedge j(b) = \vee_{i \in I}(a_i \wedge j(b))$

for all $\{a_i\}_{i \in I} \subseteq Q; a, b \in Q$.

Condition 2, is precisely the main property of a quantifier.

First of all, if A is a poset an order preserving map $F : A \to A$ is a closure map in case F satisfies:

(1) If $a \in A$ then $a \leq F(a)$.
(2) F is idempotent.

Having defined closure maps we can introduce the concept of a quantifier.

Definition 2. If L is a bounded lattice with bounds 0, 1 and $F : L \to L$ is a closure map, F will be called a quantifier on L in case F satisfies:

(1) $F(0) = 0$.
(2) If $a, b \in F$ then $F(a \wedge F(b)) = F(a) \wedge F(b)$.

There are always two special quantifiers on L. The discrete quantifier which is the identity map; the *indiscrete quantifier* quantifier defined by: $F(a) = 1$ for $a \neq 0$ and $F(0) = 0$.

If Q is an idempotent, right-sided quantale, we get a quantifier $j : Q \to Q$. Namely, $j(a) = 1\&a$ and the binary operation & is a $a \wedge j(b)$, where $a, b \in Q$.

In the rest of this article we shall work only with an idempotent, right-sided quantale. There are many examples of these quantales.

First of all, recall that a complete lattice L with bounds 0 and 1 is a *locale* if the following is satisfied:

$$(\vee_{i \in I} a_i) \wedge b = \vee_{i \in I}(a_i \wedge b) .$$

for all, $\{a_i\}_{i \in I} \subseteq L, b \in L$.

As typical example of a locale, take the open sets of a topological space X. Clearly, the open sets satisfy the last conditions.

Given an arbitrary locale L, L is an idempotent, right-sided quantale in a trivial way; the binary operation & is just $\wedge$.

The closed ideals of a C^*-algebra is also an example; here the binary operation & is the closure of the product of two ideals. The next example, needs some definitions.

Definition 3. A bounded lattice, $L = (L, \vee, \wedge, 0, 1)$ is an orthomodular lattice if there exists a unary operation $\perp : L \to L$ satisfying: for all $a, b \in L$:

(1) $a \leq b$ iff $b^{\perp} \leq a^{\perp}$.
(2) $a^{\perp\perp} = a$.
(3) $(a \vee b)^{\perp} = a^{\perp} \wedge b^{\perp}$.
(4) $a \vee a^{\perp} = 1$.
(5) $a \wedge a^{\perp} = 0$.
(6) if $a \leq b$ then $b = a \vee (a^{\perp} \wedge b)$.

We shall indicate how to induce a binary operation &, making L an idempotent, right-sided quantale. First of all, given two arbitrary elements *a, b* in an orthomodular lattice *L*, define $a\&b = (a \vee b^{\perp}) \wedge b$ and the Sasaki hook is given by the following rule: $a \to b = (a \wedge b) \vee a^{\perp}$. If we denote by $\rho_b(a) = a\&b$ and $\psi_a(b) = a \to b$ it is not hard to prove:

$$\rho_b(a) \leq c \quad \text{iff} \quad a \leq \psi_b(c). \ (*)$$

In other words, ρ_b has a right adjoint: namely, ψ_b. The map ρ_b is called the Sasaki projection. Finch (1970) considered this projection as a *binary operation* and proved the last claim. Having defined the binary operation & it is now easy to introduce the notion of compatibility and the center of an orthomodular lattice.

Definition 4. Let L be an orthomodular lattice. We say $a, b \in L$ are compatible elements (denoted by bCa) if and only if $b\&a = a \wedge b$.

Notice that whenever a, b are compatible elements it is easy to see that $a\&b = a \wedge b$ also holds. The simplest example of a pair of elements *a, b* which are compatible is whenever $b = a^{\perp}$.

Definition 5. If L is an orthomodular lattice. The center of L, denoted by $Z(L)$ is the set

$$Z(L) = \{a \in L \mid a\&b = b\&a = a \wedge b, \quad \forall b \in L\}$$

Notice that $Z(L)$ is a boolean subalgebra of L. In particular, always 0,1 belong to the center of L.

Definition 6. Let L be an orthomodular lattice. If the center is a complete sublattice of L and a is an arbitrary element of L, the central cover of a is given by

$$a(a) = \wedge\{z \in Z(L) \mid a \leq z\}.$$

The reader can see Beltrametti and Cassinelli (1981) for a discussion of the central cover, as well as some properties. Moreover, e is a *center-valued quantifier*; i.e., the image of a under e is contained in the center of L. A quantifier is not necessarily, center-valued. Indeed, suppose L is a complete orthomodular lattice and consider the following map: let z be a central element of L. Define $F(a) = (a \wedge z) \vee e(a \wedge z^{\perp})$. F is a quantifier but it is not center valued as the reader can check easily.

The central cover gives us an interesting example of an idempotent, right-sided quantale.

Proposition 1. *Suppose L is an orthomodular lattice, and the center $Z(L)$ is complete, as a sublattice, then the map $e\colon L \rightarrow L$ satisfies the conditions of Remark 1; the operation &, is defined by $a\&b = a \wedge e(b)$, making L an idempotent, right-sided quantale.*

In particular, if L is a complete orthomodular lattice all we need to generate an idempotent, right-sided quantale, is a center-valued quantifier satisfying the conditions of Remark 1.

3. SECTION II

Residuation theory is an old mathematical topic; we just want to introduce the necessary definitions for the purposes of this article. The reader can see Blyth and Janowitz (1972) for more details.

Definition 7. Let A, B two arbitrary posets and $f\colon A \rightarrow B$ a map, we say f is residuated is there is a map $f^{+} : B \rightarrow A$ satisfying:

$$f \circ f^{+} \leq 1_B, \quad 1_A \leq f^{+} \circ f \ .$$

such an $f^{+} : B \rightarrow A$ is unique and is called the residual of f.

In modern language, we just can say: f is residuated if and only if f has a right adjoint; the identity map and the composition of residuated maps are clearly residuated maps; it is not hard to show that, a residuated map f preserves suprema whenever they exist in A.

Now, thanks to the work of Foulis and Janowitz, we can present now some algebraic considerations, we think are important for the algebraic foundations of quantum mechanics. We shall present two examples of residuated endomorphisms.

The first example was considered in Section I. Indeed, take an orthomodular lattice and consider the maps ρ_b and ψ_b; it follows from $*$ of Section I that ρ_b has a right adjoint; i.e., ρ_b is a residuated map and the residual is ψ_b.

Let X be a non-empty set, and let $P(X)$ be the Boolean Algebra of all subsets of X. If R is a relation on X consider the following endomorphism $\psi_R : P(X) \rightarrow P(X)$, described by:

$$\psi_R(A) = \{y \in X | \exists x \in A, xRy\}$$

It is not hard to show that Ψ_R is residuated with residual given by

$$\psi_R^+ = i \circ \psi_{R^{op}} \circ i.$$

where $i : P(X) \to P(X)$ is the natural involution, sending any subset A of X to its complement, and R^{op} denotes the converse relation of R. Moreover, ψ_R turns out to be a residuated quantifier iff R is an equivalence relation.

Hence, given a non-empty set X and an equivalence relation R on X we get a residuated quantifier ψ_R. In particular, ψ_R preserves arbitrary suprema and clearly ψ_R satisfies the conditions of Remark 1. Hence, we get an idempotent quantale given again by the rule:

$$A\&B = A \wedge \psi_R(B)$$

For $A, B \in P(X)$. Notice that this a non-trivial example of an idempotent right-sided quantale. & preserves arbitrary suprema since $P(X)$ is a boolean algebra and ψ_R is residuated.

We can state now the main observation of this article. Recall, an orthomodular lattice L is *irreducible*, if the center of L is equal to {0, 1}. In Janowitz (1963, p. 1245), the following result can be found:

Corollary 1. *Let L be an atomic and irreducible orthomodular lattice. Then L admits only the discrete and indiscrete quantifiers.*

An example in which this situation holds is: Suppose H is an infinite dimensional Hilbert space. There is no non-trivial binary operation & defined on the lattice of closed subspaces of H, denoted by $C(H)$, making $C(H)$ an idempotent right-sided quantale.

The only thing to be noted is: $C(H)$ has trivial center; in other words, $Z[C(H)] = \{0, 1\}$ and $C(H)$ is atomic.

We close this article with some question. As the reader can see there is no hope of getting an idempotent, right-sided quantale. This was an important question since the closed ideals of a C^*-algebra precisely satisfy the conditions of being an idempotent, right-sided quantale. Also, there is an article (Rosicky, 1989) where the concept of a quantum frame is introduced. Quantum frames arise from an idempotent, right-sided quantale. Clearly, there must be a connection between the considerations we already made and this topic. We shall investigate this in a future publication.

One of the nice properties of an idempotent, right-sided quantale Q is the relation between the old operations of a lattice and the new operation &: If $a, b \in Q$ then $a \wedge b \leq a\&b \leq a$.

If L is a complete orthomodular lattice and F is a quantifier satisfying the conditions of Remark 1, then F is a center-valued quantifier and L is an idempotent, right-sided quantale, the operation & is defined by: $a\&b = a \wedge F(b)$. In particular, the fixed points of F generate a complete boolean subalgebra of L. If we start with a quantale and consider a closure operator j, satisfying the conditions of Remark 1, then the fixed points of j generate a locale, as the reader can check easily. As a consequence, we can say that from the logical point of view, we get a model of classical logic, if we start with a complete orthomodular lattice or with a model of intuitionistic logic, if we consider a quantale; in both cases, we need a quantifier F, satisfying the conditions of Remark 1.

Notice that the key point of this construction is the quantifier $F(a) = 1\&a$. Indeed, it is not hard to show that the operation $a\&b = a \wedge F(b)$ is associative, since $F(a\&F(b)) = F(a)\&F(b)$. If L is an orthomodular lattice and F is a quantifier, then F is a residuated map and therefore F preserves all the existing suprema in L, see Janowitz (1963). If we want to get a closure operator satisfying the conditions of Remark 1 we just need F is center-valued. In the case of complete lattices the closure operator defined on F must satisfies the conditions of Remark 1. These two algebraic examples can be viewed in a more general seeting: *residuated ordered semigroups*. The definition is as follows:

Definition 8. Suppose S is an ordered semigroup. S is a residuated semigroup if for each $x \in S$ the translations ρ_x, λ_x given by $\lambda_x(y) = x \circ y$ and $\rho_x(y) = y \circ x$ are residuated maps.

A quantale is just a complete lattice which is also a residuated semigroup. The preservation of arbitray suprema is trivial since the right and left translations are residuated maps; we do not need at all any completeness assumption to induce a non-trivial, non-comutative and associative binary operation on a residuated semigroup. We hope the reader understands now the following: all we need is a quantifier which is a also a residuated map.

If we consider orthomodular lattices we need a center-valued quantifier. After these considerations, there are some natural questions: can we characterize such quantifiers? There exists a general critera which allows us to consider at the same time orthomodular lattices and quantales? The answer is positive if we consider *quantic quantifiers*; one of the problems in quantum logic is that the meet operator in an orthomodular lattice L, is not a residuated map; it is not hard to show that the meet operator is residuated if and only if L is in fact a boolean algebra. However, we can replace the meet by the operator & defined in Section I, and define a quantic quantifier taking & instead of $\wedge$, in the definition of a quantifier.

This definition generalizes the concept of a quantifier and we can actually give a characterization of such maps. Not only for an orthomodular lattice, also for idempotent, right-sided quantales; in Román (2005), we present a characterization of quantic quantifiers for orthomodular lattices.

Still, there is another problem. If we consider idempotent, right-sided quantales we have a quantifier; from the point of view of residuation theory, we can generate a quantifier if we start with an idempotent, right-sided, residuated semigroup having binary meets. However, this quantifier is not necessarily a residuated map. As is well known, in the case of a Hilbert space, the situation is more complicated: we need an algebraic framework which consider the algebraic strucuture of self-adjoint operators in a Hilbert space. We want to say something about this algebraic structure.

Gudder and Greechie (1996), gave an example of a Hilbert space H with $dim(H) \geq 2$ and if $E(H)$denotes the set of all self-adjoint operators A, satisfying $0 \leq A \leq I$ then $E(H)$ *IS NOT A LATTICE*. The partial order is defined by setting: $A \leq B$ if $\langle Ax, x\rangle \leq \langle Bx, x\rangle$, for all $x \in H$.

Hence, it is not a good approach to start with a *complete lattice* if we are really interested in the algebraic foundations of quantum mechanics; we believe that residuated ordered semigroups, is the right approach.

ACKNOWLEDGMENTS

I express my sincere thanks to Prof. R. Greechie and Prof. M. F. Janowitz. The electronic messages I had with Prof. Janowitz were very stimulating and encouraged me to solve this problem. Part of this work was realized while I spent my sabbatical year as a research visitor at Louisiana Tech University. Many thanks to Prof. Greechie, who made this invitation possible. This research was supported by a BECA SABÁTICA from la Dirección General de Asuntos del Personal Académico de la UNAM. Many thanks to the referees for their suggestions.

REFERENCES

Beltrametti, E. G. and Cassinelli, G. (1981). The logic of quantum mechanics. *Encyclopedia of Mathematics and its Applications* **15**.

Birkhkoff, G. and von Neumann, J. (1936). The logic of quantum mechanics. *Annals of Mathematics*, Ser. 2 **37**, 823–246.

Blyth, S. T. and Janowitz, F. M. (1972). *Residuation Theory*, Vol. 102. Pergamon Press.

Borceux, F. and van den Bossche, G. (1986). Quantales and their sheaves. *Order* **3**, 61–87.

Finch P. D. (1970). Quantum logic as an implication algebra. *Bulletin of the Australian Mathematical Society* **1**, 101–106.

Gudder, S. and Greechie, R. (1996). Effect algebra counterexamples. *Mathematica Slovaca* **46**(4), 317–325.

Halmos, P. (1962). *Algebraic Logic*, Chelsea Publishing Company, New York.

Janowitz, M. F. (1963). Quantifiers and orthomodular lattices. *Pacific Journal of Mathematics* **13**, 1241–1249.

Mackey, W. G. (1957). Quantum Mechanbics and Hilbert space, Herbert Ellsworth Memorial Paper No. 6. *The American Mathematical Monthly* **64**(8), 45–57.

Roman, L. (2005). A characterization of quantic quantifiers for orthomodular lattices. *Theory and Application of Categories* (submitted).

Rosicky, J. (1989). Multiplicative Lattices and C^*-algebras. *Cahiers de Topologie et Geométrie Diferentielle Categoriques* **XXX-2**, 95–110.